1 Graph Theory

1. **Graph Counting I:** A **complete** graph is a graph where between any two nodes a, b there is an edge. How many edges are in an undirected complete graph with n nodes?

2. **Graph Counting II:** What is the minimal number of bits required to encode the adjacency matrix of any directed, unweighted graph with n nodes?

3. **Graphs and Paths I:** Write an algorithm that when given a graph with a countably infinite number of vertices will output a shortest path between two nodes.

   Hint: Depth first search will not work. Why?

4. **Graphs and Paths II:** Write an algorithm that when given a graph (V, E) outputs a Eulerian circuit. An optimal algorithm is $O(|E|)$. Prove the correctness of your algorithm.

5. **A Foray into the Tropical I:** The *tropical semiring* is the usual set of real numbers $\mathbb{R}$ along with ∞, except its addition $\oplus$, and multiplication $\otimes$ are different:

   $$a \oplus b = \min(a, b)$$

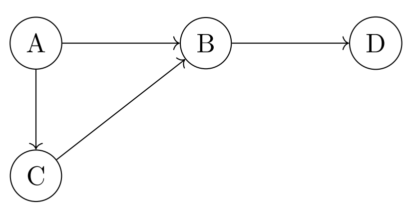
   and

   $$a \otimes b = a + b$$

   So $3 \oplus 5 \otimes 2 = 3 \oplus 7 = 3$

   We can use these operations to define tropical matrix multiplication – which works the same as regular matrix multiplication but we use the tropical operations.

   We can use tropical matrix multiplication to solve the shortest path problem. We can turn a graph:

   Into a tropical matrix:

   $$\begin{bmatrix} 0 & 1 & 1 & \infty \\ \infty & 0 & \infty & 1 \\ \infty & 1 & 0 & \infty \\ \infty & \infty & \infty & 1 \end{bmatrix}$$

   By placing ∞ in the entries with no corresponding edges, 0 in the self loops, and 1 (or the weight of the edge) on the edges.

(a) Show that if A is the tropical matrix representing a graph, then A^n (using tropical matrix multiplication) has in entry i, j the length of the shortest path between i and j that uses at most n steps, and that $\lim_{n\to\infty} A^n$ has as entries the shortest paths between any two nodes.

(b) How can you extend this to a graph with arbitrary weights?

(c) How could we compute what $\lim_{n\to\infty} A^n$ is?

6. **A Foray into the Tropical II:** The assignment problem is the following: You have a set of tasks T and a set of workers W and a cost to assigning a worker i to a task t: $c(i, t)$. Each worker can be given at most a single task. What is the minimum cost possible to complete all tasks?

   This can be represented as a graph problem where you have nodes for each task and worker, and an edge between each worker and task with weight $c(i, t)$. The problem becomes the problem of picking a subgraph of this graph that looks like a bunch of pairings of workers and tasks which has minimal total weight.

   This can also be solved tropically. You can set up a matrix:

$$\begin{bmatrix} c(1,1) & c(1,2) & c(1,3) & c(1,4) \\ c(2,1) & c(2,2) & c(2,3) & c(2,4) \\ c(3,1) & c(3,2) & c(3,3) & c(3,4) \\ c(4,1) & c(4,2) & c(4,3) & c(4,4) \end{bmatrix}$$

   If you take the tropical *permanent* of this matrix, it gives you the minimum cost possible.

   The *permanent* is the determinant if you do not negate anything.

   Explain why this works and use this fact to find an efficient algorithm to find the tropical permanent of a matrix

7. **Max Flow:** The Ford-Fulkerson algorithm is not the most efficient algorithm to find the max flow. Find an improvement to Ford-Fulkerson to make its runtime polynomial in V and E

   Hint: Consider how you can choose the path in step 2

   Hint 2: Modifying that step can create the *Edmonds-Karp algorithm.* Further improvements get you *Dinic's algorithm.*

2 Automata

2.1 Exercises

1. **Finite Automata I:** Create a finite automata that recognizes the language: $\{w : w \text{ has an odd number of 0's}\}$ where strings are expressed in binary.

2. **Finite Automata II:** Create a finite automata that recognizes the language: $\{w : w$ has an even number of 0's and all three character substrings are 001 or 101 $\}$ where strings are expressed in binary.

3. **Finite Automata III:**

   (a) Prove that the class of regular languages is closed under union.

   (b) Prove that the class of regular languages is closed under compliment.

   (c) Prove that if a class of sets is closed under unions and compliments then it is closed under intersection.

4. **Finite Automata IV:** Use the pumping lemma to prove that the language $\{0^n 1^n\}$ is not expressible by a DFA.

5. **Finite Automata V:** Prove the language $\{0^a 1^b | a \neq b\}$ is not expressible by a DFA.

   Hint: Examine problems 3 and 4 and use the fact that $\{0^a 1^b\}$ is expressible by a DFA without the restriction that $a \neq b$.

6. **Turing Computability I:** Prove that $HALT$ is Turing recognizable.

7. **Turing Computability II:** Prove that $HALT$ is not decidable.

   Hint: Assume it is. Then there is a Turing machine that decides it. Use that Turing machine to construct a Turing machine that would cause something akin to the Barber paradox.

8. **Turing Reductions I:** Examine the problem ES: Given a Turing machine M, accept if and only if the language of M is empty.

   Prove that this problem is undecidable by finding a reduction from $HALT$ to ES. (If EQ were decidable and such a reduction existed then $HALT$ would be which we know it is not.)

9. **Turing Reductions II:** Examine the problem EQ: Given two Turing machines M, N accept if and only if the language of M is the same as N

   Prove that this problem is undecidable by finding a reduction from $HALT$ to EQ.